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CONJUGATE SHEAR FLOWS OF A WEAKLY STRATIFIED FLUID

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This paper studies the problem of pairs of horizontal shear flows of weakly stratified fluids with identical mass, momentum, and energy fluxes. The initial problem is reduced to a system of two scalar equations for the main- and perturbed-flow parameters by using bifurcation methods. The existence conditions for nontrivial branches of conjugate flows close to the main flow are investigated.

Key words: *conservation laws, internal waves, stratified fluid, conjugate flows, bifurcation of solutions of differential equations.*

Introduction. The problem of describing pairs of shear flows with identical mass, momentum, and energy fluxes arises in studies of two-dimensional stationary waves in a stratified fluid. Following [1], pairs of such horizontal flows are called conjugate. In particular, the horizontal flow ahead of the front of a smooth bore type wave is conjugate to the flow behind the front of this wave [2, 3]. The pairs of flows arising ahead of the front of an internal solitary plateau type wave and in its middle part are also conjugate flows [4, 5].

Analytical existence conditions for conjugate flows in a continuously stratified fluid are obtained in [3, 6], where the problem of conjugate flows was reduced to a two-dimensional system of bifurcation equations by using the Lyapunov–Schmidt scheme. By analysis of this system, sufficient conditions for the existence of a locally unique branch of flows conjugate to a uniform flow are derived and examples of nonuniqueness of the solutions are constructed. Nonuniqueness was also noted in [4], where flows conjugate to a uniform flow of a stratified fluid were investigated numerically.

In the present paper, the existence conditions for the solutions obtained in [6] are specified and generalized using the approach proposed in that paper. In particular, a more detailed investigation of the local properties of the solution whose existence was established in [6] leads to the necessity of formulating alternative conditions for the existence of a nontrivial solution of the problem. In addition, in the present paper, shear main flow is studied, allowing a generalization of existing results.

1. Formulation of the Problem. The plane stationary flow of an inhomogeneous incompressible fluid in a layer $\{-\infty < x < \infty,\ 0 < y < h\}$ enclosed between an even bottom ($y = 0$) and a rigid cover ($y = h$) is described by the Euler equations

$$\rho(UU_x + VU_y) + p_x = 0, \qquad \rho(UV_x + VV_y) + p_y = -\rho g,$$
$$U_x + V_y = 0, \qquad U\rho_x + V\rho_y = 0 \tag{1.1}$$

with the nonpenetration boundary conditions $V = 0$ ($y = 0, y = h$). Here ρ is the density, (U, V) is the velocity, p is the pressure, and g is the acceleration due to gravity. The conservation of density along the streamlines provides the existence of a functional dependence of the density ρ on the stream function ψ. In view of this, after elimination of the pressure by using the Bernoulli integral

$$\rho|\nabla\psi|^2/2 + \rho gy + p = B(\psi) \tag{1.2}$$

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system (1.1) reduces to one quasilinear elliptic equation of the second order (the Dubreil-Jacotin–Long [equation [7])

$$\rho(\psi)\Delta\psi + \rho_\psi(\psi)((\nabla\psi)^2/2 + gy) = B_\psi(\psi). \tag{1.3}$$

Here $B(\psi)$ is a Bernoulli function; the subscript denotes differentiation with respect to the corresponding variable.

We assume that the density profile $\rho = \rho_0(y/h, \sigma)$ and the stream function of the shear flow $\psi = \psi_0(y/h)$ are known:

$$\rho_0(Y, \sigma) = \rho_*(1 + \sigma\rho_1(Y) + \sigma^2\rho_2(Y, \sigma)), \qquad \psi_0(Y) = ch\psi_1(Y). \tag{1.4}$$

Here ρ_* is the characteristic density scale, σ is a small Boussinesq parameter related to the characteristic buoyancy frequency N_0 by the formula $\sigma = N_0^2 h/g$. The dimensionless functions ρ_1 and ρ_2 specify the background density profile and the fine stratification structure, respectively [8]. In turn, the stream function ψ_0 is characterized by the main-flow velocity c at $y = 0$ and the dimensionless function ψ_1. The flow given by relation (1.4) will be called the main flow.

Next, it is assumed that, everywhere in their domain of definition, the functions $\rho_1(Y) \in C^4[0, 1]$, $\rho_2(Y, \sigma) \in C^4([0, 1] \times [0, \sigma_0])$ and $\psi_1(Y) \in C^4[0, 1]$ satisfy the inequalities

$$\rho_0 > 0, \qquad \rho_{0Y} < 0, \qquad \rho_{1Y} < 0, \qquad \psi_{1Y} \neq 0. \tag{1.5}$$

The constraints imposed on the density guarantee the stability of the stratification, and the condition imposed on the stream function guarantees the absence of return flows in the main flow (this requirement provides invertibility of the function ψ_0). Conditions (1.5) allow one to determine the form of the functions $\rho(\psi)$ and $B_\psi(\psi)$ included in Eq. (1.3). Indeed, the inversion of the dependence $\psi = \psi_0(y)$ gives the relationship $y = y_0(\psi)$ between the stream function and the variable y in the main flow, which allows one to determine the dependence of the density on the stream function and [by virtue of Eq. (1.3)] the form of the function $B_\psi(\psi)$:

$$\rho(\psi) = \rho_0(y_0), \qquad B_\psi(\psi) = \rho(\psi)\psi_{0yy}(y_0) + \rho_\psi(\psi)(\psi_{0y}^2(y_0)/2 + gy_0). \tag{1.6}$$

Here the argument ψ of the functions $y_0(\psi)$ is omitted for brevity.

The coincidence of the distributions of the fluid density and Bernoulli function along the streamlines in the main and conjugate flows provides coincidence of the mass and energy fluxes of these flows. The equality of the momentum fluxes leads to an additional relation that should be taken into account in studying the conjugate flows arising in stationary wave configurations such as a solitary wave or a smooth bore [3, 4]. For plane fluid flow with horizontal streamlines given by the stream function $\psi(y)$, the momentum flux $F_{\rm imp}(\psi)$ can be written as

$$F_{\rm imp}(\psi) = \int_0^h (\rho(\psi)\psi_y^2 + p)\,dy.$$

Eliminating the pressure p by virtue of (1.2), the difference of the momentum fluxes in the main and conjugate flows can be written as

$$\int_0^h \Big(\frac{\rho(\psi)\psi_y^2}{2} - gy\rho(\psi) + B(\psi)\Big)dy - \int_0^h \Big(\frac{\rho_0(y)\psi_{0y}^2}{2} - gy\rho_0(y) + B(\psi_0)\Big)dy = 0. \tag{1.7}$$

Thus, the conjugate flow is described by the system of equations containing the one-dimensional version of the Dubreil-Jacotin–Long equation (1.3) with the function $B_\psi(\psi)$ specified by formula (1.6) and the integral relation (1.7).

We transform to dimensionless variables using the quantity h as the characteristic scale for the variable y and the function y_0, the quantity ch as the scale for the stream functions ψ and ψ_0, the quantity ρ_* as the density scale, and $\rho_* c^2$ as the scale for the Bernoulli function B. Using the previous notation for all dimensionless quantities, we obtain the nonlinear ordinary differential equation

$$\rho(\psi)(\psi_{yy} - \psi_{0yy}(y_0)) + \rho_\psi(\psi)((\psi_y^2 - \psi_{0y}^2(y_0))/2 + \lambda\sigma^{-1}(y - y_0)) = 0 \tag{1.8}$$

on the interval $y \in (0, 1)$ with the boundary conditions $\psi(0) = 0$, $\psi(1) = Q$ [$Q = \psi_0(1)$ is the fluid flux in the main flow] and the additional integral condition

$$\int_0^1 \Big(\frac{\rho(\psi)\psi_y^2}{2} - \lambda\sigma^{-1} y\rho(\psi) + B(\psi)\Big) dy - \int_0^1 \Big(\frac{\rho_0(y)\psi_{0y}^2}{2} - \lambda\sigma^{-1} y\rho_0(y) + B(\psi_0)\Big) dy = 0, \tag{1.9}$$

where $\lambda = \sigma gh/c^2$ is the densimetric (density) Froude number.

Equations (1.8) and (1.9) contain nonlinearity of general form given by the functions $\rho(\psi)$ and $y_0(\psi)$. To simplify the further consideration, we transform to the semi-Lagrangian variables [ψ and $y(\psi)$], thus obtaining equations with fractional rational nonlinearity. This is possible by virtue of the postulated absence of return flows. In this case, the transformed equations will be considered on the interval $\psi \in [0, Q]$. In view of the relations $\psi_y = 1/y_\psi$ and $\psi_{yy} = -y_{\psi\psi}/y_\psi^3$, Eq. (1.8) becomes

$$\Big(\rho\, \frac{y_{0\psi}^2 - y_\psi^2}{2y_\psi^2 y_{0\psi}^2}\Big)_\psi + \lambda(y - y_0)\tilde{\rho}_\psi = 0,$$

where $\tilde{\rho}(\psi) = \sigma^{-1}\rho(\psi)$; the argument ψ of the functions $\rho(\psi)$ and $\tilde{\rho}(\psi)$ is omitted for brevity. To transform the momentum flux integral (1.9), we perform the change of variables $y = y(\psi)$ in the first term and $y = y_0(\psi)$ in the second. As a result, the integral relation becomes

$$\int_0^Q \Big(\rho\, \frac{y_{0\psi} - y_\psi}{2y_\psi y_{0\psi}} - \lambda(yy_\psi - y_0 y_{0\psi})\tilde{\rho} + B(\psi)(y_\psi - y_{0\psi})\Big) d\psi = 0. \tag{1.10}$$

The solution of the formulated problem is sought in the vicinity of the main flow given by the eigenfunction $y_0(\psi)$:

$$y(\psi) = y_0(\psi) + w(\psi).$$

As a result, we have the following system of equations for the perturbation $w(\psi)$:

$$F(w; \lambda, \sigma) \stackrel{\text{def}}{=} -\Big(\frac{\rho w_\psi}{y_{0\psi}^3}\Big)_\psi + \lambda\tilde{\rho}_\psi w + \Big(\rho\, \frac{3y_{0\psi}w_\psi^2 + 2w_\psi^3}{2y_{0\psi}^3 (y_{0\psi} + w_\psi)^2}\Big)_\psi = 0; \tag{1.11}$$

$$w(0) = w(Q) = 0; \tag{1.12}$$

$$\int_0^Q \Big(\frac{\rho w_\psi^2}{2y_{0\psi}^2 (y_{0\psi} + w_\psi)} + \frac{\lambda}{2}\tilde{\rho}_\psi w^2\Big) d\psi = 0. \tag{1.13}$$

The integral relation (1.10) in the final form (1.13) is obtained by integration by parts in the term containing $B(\psi)$, taking into account boundary conditions (1.12) and using expression (1.6) for the derivative $B_\psi(\psi)$ of the Bernoulli function.

2. Bifurcation Problem. It is easy to see that the solution $w(\psi) \equiv 0$ satisfies system (1.11)–(1.13) for any λ and σ. Thus, the problem of conjugate flows can be treated as the problem of bifurcation of the trivial solution of Eqs. (1.11) and (1.12) with the additional integral relation (1.13). Using the Lyapunov–Schmidt scheme (see, for example, [9]), which allows the description of the behavior of solutions in the vicinity of bifurcation points, we determine the following functional spaces:

$$\mathbb{E} = \{v \in C^2[0, Q]\colon v(0) = v(Q) = 0\}, \qquad \mathbb{F} = C[0, Q].$$

The nonlinear differential operator of Eq. (1.11) will be considered as a map $F(w(\psi); \lambda, \sigma)$: $\mathbb{E} \times \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{F}$. We seek conjugate flows close to the main flow. Then, the condition of smallness of the perturbation w allows the use of some properties of the linear part of the operator of problem (1.11):

$$L(\lambda, \sigma)\langle w\rangle \stackrel{\text{def}}{=} -(\rho w_\psi / y_{0\psi}^3)_\psi + \lambda\tilde{\rho}_\psi w.$$

We recall that the initial problem contains a natural small parameter of weak stratification $\sigma > 0$. Therefore, we consider the bifurcation points for the limiting equation (1.11) for $\sigma = 0$. For this equation, such points can be only the eigenvalues λ_n of the Sturm–Liouville problem

$$-(\varphi_\psi / y_{0\psi}^3)_\psi + \lambda\rho_{1\psi}\varphi = 0, \qquad \varphi(0) = \varphi(Q) = 0, \tag{2.1}$$

where the function $\rho_1 = \rho_1(y_0(\psi))$ is defined in (1.4). If conditions (1.5) are satisfied, this problem has a countable family of single real eigenvalues $\{\lambda_n\colon n \in \mathbb{N}\}$ (numbered in increasing order) and the family of their corresponding eigenfunctions $\{\varphi_n\colon n \in \mathbb{N}\}$ [7].

Let us fix the minimum eigenvalue λ_0 and its corresponding eigenfunction $\varphi_0(\psi) \in \mathbb{E}$ normalized in $L_2[0, Q]$. The space $\mathbb{E}$ is represented as the direct sum of subspaces $\mathbb{E} = \ker P \oplus \operatorname{im} P$ with the projector $P\colon \mathbb{E} \mapsto \mathbb{E}$ given by the formula

$$P\langle v\rangle = \varphi_0 \int\limits_0^Q \varphi_0(\psi) v(\psi)\, d\psi. \tag{2.2}$$

In view of the self-conjugation of the operator $L(\lambda_0, 0)$ it can be concluded that the nonuniform problem

$$L(\lambda_0, 0)\langle w\rangle = f, \qquad f \in \mathbb{F}$$

has a unique solvability condition of the form $H\langle f\rangle = 0$ with the projector H onto the defect subspace given by the same formula (2.2). Similarly, the space $\mathbb{F}$ is represented as $\mathbb{F} = \ker H \oplus \operatorname{im} H$. We note that the eigenfunctions φ_n form a basis in $L_2[0, Q]$ which is orthogonal to the scalar product with weight $\rho_{1\psi}$.

Equation (1.11) is written as

$$L(\lambda_0, 0)\langle w\rangle = R(w; \lambda, \sigma), \tag{2.3}$$

where $R(w; \lambda, \sigma) = L(\lambda_0, 0)\langle w\rangle - F(w, \lambda, \sigma)$. The operator $R(w; \lambda, \sigma)$ is small but not lower than the second order for the set of variables w, $\lambda - \lambda_0$, and σ. Indeed, the main-order term of the expression for $R(w; \lambda, \sigma)$ contains only products of the form $\sigma w_{\psi\psi}$, σw_ψ, $(\lambda - \lambda_0)w$, w_ψ^2, and $w_{\psi\psi} w_\psi$, as follows from the form of the left side of (1.11) and the representation of $L(\lambda, \sigma)\langle\,\cdot\,\rangle$ for $\lambda = \lambda_0$ and $\sigma = 0$.

Following the standard Lyapunov–Schmidt scheme, we seek the solution of the problem in the form

$$w = b\varphi_0 + u, \tag{2.4}$$

where $b \in \mathbb{R}$ is the amplitude parameter and the function $u \in \ker P$ satisfies the condition $P\langle u\rangle = 0$. In view of the expansion of the space $\mathbb{F}$ generated by the projector H, Eq. (2.3) can be written in projections onto the subspaces $\ker H$ and $\operatorname{im} H$:

$$L(\lambda_0, 0)\langle u\rangle = (I - H)\langle R(b\varphi_0 + u; \lambda, \sigma)\rangle; \tag{2.5}$$

$$0 = H\langle R(b\varphi_0 + u; \lambda, \sigma)\rangle. \tag{2.6}$$

Statement 1. *There exists a smooth map* $u(\lambda, \sigma, b) : \mathbb{R}^3 \mapsto \ker P$ *which is defined in some vicinity of the point* $(\lambda_0, 0, 0)$ *and transform Eq.* (2.5) *to identity, with the equality* $u(\lambda, \sigma, 0) = 0$ *being satisfied for all* λ *and* σ *from the domain of definition. In addition,* $u(\lambda_0, 0, b) = O(b^2)\Big|_{b\to 0}$.

The existence of the indicated map follows directly from the theorem of implicit maps applied to Eq. (2.5) because the operator L considered as a map $L(\lambda_0, 0)\langle\cdot\rangle : \ker P \to \ker H$ is a continuous invertible operator. The above estimates of the order of smallness of the function u are obtained as a consequence of the boundedness of the operator L^{-1} in view of the boundedness of the projector H and the order of smallness of the residue $R(b\varphi_0 + u; \lambda, \sigma)$.

Because of the smoothness of the map $u(\lambda, \sigma, b)$ and its properties indicated in Statement 1, a smooth map $\tilde{u}$ exists such that

$$u(\lambda, \sigma, b) = b\tilde{u}(\lambda, \sigma, b). \tag{2.7}$$

With the function $\tilde{u}$ introduced in such a manner, Eq. (2.6) written in the form

$$H\langle R(b\varphi_0 + b\tilde{u}(\lambda, \sigma, b); \lambda, \sigma)\rangle = 0 \tag{2.8}$$

is an implicitly specified scalar bifurcation equation for the three parameters λ, σ, and b. This equation describes the behavior of all branches of the solution of Eqs. (1.11) and (1.12) in the vicinity of the bifurcation point: any small solution of Eq. (2.8) of the form $\lambda(\sigma, b)$ can be put in correspondence to the small solution of Eqs. (1.11) and (1.12) $w = b\varphi_0 + u(\lambda(\sigma, b), \sigma, b)$.

3. Analysis of the Scalar System of Equations. For the description of the small conjugate flows, it remains to take into account the condition of coincidence of the horizontal momentum fluxes (1.13). Taking into account the form of solution (2.4) and the smallness property of the functions u in (2.7) and substituting (2.4) into (1.11) and (1.13), we obtain the following real system of equations describing conjugate flows close to the main flow:

$$bf(\lambda,\sigma,b)=0,\qquad b^2 l(\lambda,\sigma,b)=0. \tag{3.1}$$

Here

$$f(\lambda,\sigma,b)=\int_0^Q \varphi_0\Big[-\Big(\frac{\rho(\varphi_{0\psi}+\tilde u_\psi)}{y_{0\psi}^3}\Big)_\psi+\lambda\tilde\rho_\psi(\varphi_0+\tilde u)+\Big(\rho\,\frac{3by_{0\psi}(\varphi_{0\psi}+\tilde u_\psi)^2+2b^2(\varphi_{0\psi}+\tilde u_\psi)^3}{2y_{0\psi}^3(y_{0\psi}+b\varphi_{0\psi}+b\tilde u_\psi)^2}\Big)_\psi\Big]d\psi; \tag{3.2}$$

$$l(\lambda,\sigma,b)=\int_0^Q\Big(\frac{\rho(\varphi_{0\psi}+\tilde u_\psi)^2}{2y_{0\psi}^2(y_{0\psi}+b\varphi_{0\psi}+b\tilde u_\psi)}+\frac{\lambda}{2}\,\tilde\rho_\psi(\varphi_0+\tilde u)^2\Big)d\psi. \tag{3.3}$$

Because we seek nontrivial conjugate flows (i.e., $w\neq 0$), by setting $b\neq 0$ we transform from system (3.1) to the system

$$f(\lambda,\sigma,b)=0,\qquad l(\lambda,\sigma,b)=0, \tag{3.4}$$

which corresponds to the separation of the trivial branch of the solution. A pair of smooth functions [$b(\sigma)$ and $\lambda(\sigma)$] for which both equations of system (3.4) become identity in some vicinity of zero and $b(0)=0$ and $\lambda(0)=\lambda_0$ will be called a small solution of system (3.4). Below, we study conditions at which system (3.4) admits small solutions.

Statement 2. *System* (3.4) *has a small solution of the form*

$$b\equiv 0,\qquad \lambda=\tilde\lambda(\sigma) \tag{3.5}$$

with some smooth function $\tilde\lambda(\sigma)$.

Proof. In the first equation in (3.4), we set $b=0$. Then, in view of relations (3.2) and (3.3), we obtain

$$f(\lambda,\sigma,0)=\int_0^Q \varphi_0 L(\lambda,\sigma)\langle\varphi_0+\tilde u\rangle\,d\psi. \tag{3.6}$$

Performing integration by parts in expression (3.3) for l, for $b=0$ we obtain

$$l(\lambda,\sigma,0)=\frac12\int_0^Q(\varphi_0+\tilde u)\Big[-\Big(\frac{\rho(\varphi_{0\psi}+\tilde u_\psi)}{y_{0\psi}^3}\Big)_\psi+\lambda\tilde\rho_\psi(\varphi_0+\tilde u)\Big]d\psi$$

$$=\frac12\int_0^Q\varphi_0 L(\lambda,\sigma)\langle\varphi_0+\tilde u\rangle\,d\psi+\frac12\int_0^Q\tilde u L(\lambda,\sigma)\langle\varphi_0+\tilde u\rangle\,d\psi.$$

Let us show that the last term on the right side is equal to zero for any λ and σ. Relation (2.5) written in the equivalent form

$$(I-H)\langle F(b\varphi_0+b\tilde u(\lambda,\sigma,b);\lambda,\sigma)\rangle=0$$

is identity in some vicinity of the point $(\lambda_0,0,0)$. Differentiation of this identity with respect to the variable b at the point $b=0$ yields

$$\frac{\partial}{\partial b}\Big[(I-H)\langle F(b\varphi_0+b\tilde u(\lambda,\sigma,b);\lambda,\sigma)\rangle\Big]\Big|_{b=0}=(I-H)\langle L(\lambda,\sigma)\langle\varphi_0+\tilde u\rangle\rangle=0.$$

The last equality implies that $L(\lambda,\sigma)\langle\varphi_0+\tilde u\rangle\in\ker(I-H)$, i.e., $L(\lambda,\sigma)\langle\varphi_0+\tilde u\rangle=c\varphi(\psi)$, where c depends only on the parameters λ and σ. Because $\tilde u\in\ker H$, we have

$$\int_0^Q\tilde u L(\lambda,\sigma)\langle\varphi_0+\tilde u\rangle\,d\psi=c(\lambda,\sigma)\int_0^Q\tilde u\varphi_0\,d\psi=0.$$

Hence, for any values of the parameters λ and σ in some vicinity of the point $(\lambda_0,0)$, the following relation holds:

$$l(\lambda, \sigma, 0) = f(\lambda, \sigma, 0)/2. \tag{3.7}$$

Because Eqs. (3.4) are linearly dependent for $b = 0$, to complete the proof it suffices to show that there exists a smooth function $\lambda(\sigma)$ for which the equation $f(\lambda, \sigma, 0) = 0$ becomes identity. For this, we use the implicit-function theorem. By virtue of relation (3.6), we have $f(\lambda_0, 0, 0) = 0$; therefore, it only remains to verify that the derivative $f_\lambda(\lambda_0, 0, 0)$ does not vanish. Taking into account that the eigenfunction φ_0 is orthogonal to the image of the operator $L(\lambda_0, 0)$ and using the equality $\tilde{u}(\lambda_0, 0, 0) = 0$ for $b = 0$, from (3.2) we obtain

$$f_\lambda(\lambda_0, 0, 0) = \int_0^Q \varphi_0\Big[-\Big(\frac{\tilde{u}_{\lambda\psi}}{y_{0\psi}^3}\Big)_\psi + \lambda_0\rho_{1\psi}\tilde{u}_\lambda + \rho_{1\psi}(\varphi_0 + \tilde{u})\Big]d\psi = \int_0^Q \varphi_0\Big(L(\lambda_0, 0)\langle\tilde{u}_\lambda\rangle + \rho_{1\psi}\varphi_0\Big)\, d\psi = \chi,$$

where

$$\chi \stackrel{\text{def}}{=} \int_0^Q \rho_{1\psi}\varphi_0^2\, d\psi \neq 0 \tag{3.8}$$

by virtue of the stratification stability condition (1.5).

Thus, system (3.4) admits a trivial solution for $b(\sigma) \equiv 0$ even after the separation of the trivial branch in (3.1). The following statement represents the conditions under which system (3.4) possesses a unique small solution.

Statement 3. *If the condition*

$$\mu \stackrel{\text{def}}{=} \int_0^Q \frac{\varphi_{0\psi}^3}{y_{0\psi}^4}\, d\psi \neq 0 \tag{3.9}$$

is satisfied, there exists a unique pair of smooth functions $\lambda(\sigma)$ and $b(\sigma)$ that specify the small solution of system (3.4).

Proof. To prove this statement, we again use the implicit-function theorem. As noted in the proof of Statement 2, the point $(\lambda_0, 0, 0)$ satisfies both equations of system (3.4). Thus, the local solvability of this system of equations is characterized by the determinant of the matrix

$$M = \begin{pmatrix} f_b(\lambda_0, 0, 0) & f_\lambda(\lambda_0, 0, 0) \\ l_b(\lambda_0, 0, 0) & l_\lambda(\lambda_0, 0, 0) \end{pmatrix}.$$

The derivative $f_\lambda(\lambda_0, 0, 0)$ was already calculated in the proof of Statement 2, and the derivative $l_\lambda(\lambda_0, 0, 0)$ is expressed from relation (3.7):

$$f_\lambda(\lambda_0, 0, 0) = \chi, \qquad l_\lambda(\lambda_0, 0, 0) = \chi/2.$$

The derivatives of the functions with respect to the parameter b can be obtained directly: By virtue of relations (3.2), taking into account that $\tilde{u}(\lambda_0, 0, 0) = 0$, and performing integration by parts in the second term, we have

$$f_b(\lambda_0, 0, 0) = \int_0^Q \varphi_0\Big[L(\lambda_0, 0)\langle\tilde{u}_b\rangle + \frac{\partial}{\partial b}\Big(\frac{3by_{0\psi}(\varphi_{0\psi} + \tilde{u}_\psi)^2 + 2b^2(\varphi_{0\psi} + \tilde{u}_\psi)^3}{2y_{0\psi}^3(y_{0\psi} + b\varphi_{0\psi} + b\tilde{u}_\psi)^2}\Big)_\psi\Big]d\psi = -\frac{3}{2}\int_0^Q \frac{\varphi_{0\psi}^3}{y_{0\psi}^4}\, d\psi.$$

Similarly, for the derivative of the functions $l(\lambda, \sigma, b)$ with respect to the variable b, from (3.3) we obtain

$$l_b(\lambda_0, 0, 0) = \int_0^Q \Big(\frac{\varphi_{0\psi}\tilde{u}_{b\psi}}{y_{0\psi}^3} + \lambda_0\rho_{1\psi}\varphi_0\tilde{u}_b - \frac{\varphi_{0\psi}^3}{2y_{0\psi}^4}\Big)\, d\psi = \int_0^Q \Big(\varphi_0 L(\lambda_0, 0)\langle\tilde{u}_b\rangle - \frac{\varphi_{0\psi}^3}{2y_{0\psi}^4}\Big)\, d\psi = -\frac{1}{2}\int_0^Q \frac{\varphi_{0\psi}^3}{y_{0\psi}^4}\, d\psi.$$

From this, the determinant of the matrix M is expressed as

$$\det M = -\mu\chi/4$$

[χ and μ are given by formulas (3.8) and (3.9), respectively]. As noted above, $\chi \neq 0$; therefore, condition (3.9) provides the existence of the unique small solution of system (3.4).

Thus, if a trivial branch of solutions (3.5) necessarily exists, Statement 3 establishes the uniqueness of this branch in the case where the determinant of the matrix M is different from zero. Statements 1–3 lead to the following statement.

Statement 4. *The condition*

$$\int_0^Q \frac{\varphi_{0\psi}^3}{y_{0\psi}^4}\,d\psi = 0 \tag{3.10}$$

is necessary for the existence of branches of conjugate flows different from the main flow.

For the case of a uniform main flow, a statement equivalent to Statement 3 is proved in [6]. However, in that paper, the properties of the solution branch distinguished by condition (3.9) were not analyzed since the primary goal was to study the nonuniqueness of the solution. Thus, Statement 4 significantly supplements the interpretation of the existence conditions for small conjugate flows obtained in [6].

We also note that, in the case of uniform main flow, condition (3.10) can be obtained by using the form of the momentum flux integral employed in the numerical solution of the conjugate flow problem in [4].

4. Equivalent Formulation of the Problem in Eulerian Variables. The above line of reasoning can be used to solve the problem in the initial variables [$\psi(y)$ and y]. In this case, condition (3.10) is written in the more complex form

$$\int_0^1 \Big[\frac{2\lambda_0}{(\psi_{0y})^{3/2}}\Big(\frac{\rho_{1y}}{(\psi_{0y})^{3/2}}\Big)_y + \frac{1}{\psi_{0y}}\Big(\frac{\psi_{0yyy}}{\psi_{0y}}\Big)_y\Big]\xi_0^3\,dy = 0. \tag{4.1}$$

Here ξ_0 is the eigenfunction of the Sturm–Liouville problem which is generated for $\sigma = 0$ by the linear part of the operator of problem (1.8):

$$\xi_{yy} - \Big(\lambda\,\frac{\rho_{1y}}{(\psi_{0y})^2} + \frac{\psi_{0yyy}}{\psi_{0y}}\Big)\xi = 0, \qquad \xi(0) = \xi(1) = 0. \tag{4.2}$$

Conditions (3.10) and (4.1) are equivalent if they are considered as constraints imposed on the parameters of the main flow (1.4). The proof of the equivalence is based on the relation between the eigenfunctions of the Sturm–Liouville problem (2.1) and (4.2):

$$\varphi_0(\psi) = -\xi_0(y_0(\psi))y_{0\psi}(\psi).$$

In fact, this relation implies that the eigenfunction φ_0 is the image of the eigenfunction ξ_0 obtained by the action of the linear par of the transformation of variables $[\psi(y), y) \to (y(\psi), \psi]$. In this case, the eigenvalues coincide.

Condition (4.1) is more constructive in the sense that it allows one to obtain velocity and density profiles for which this condition is obviously satisfied. In particular, solving the system of differential equations

$$\psi_{0yyy} = C_1\psi_{0y}, \qquad \rho_{1y} = C_2(\psi_{0y})^{3/2} \qquad (C_{1,2} = \text{const})$$

and choosing the remaining integration constants so that the average density gradient has order σ, as the dimensionless parameters of the main flow for $C_1 = 0$, we obtain

$$\psi_0(y) = y + \frac{\gamma y^2}{2}, \qquad \rho_0(y) = 1 - \frac{(1+\gamma y)^{5/2} - 1}{(1+\gamma)^{5/2} - 1}\,\sigma + O(\sigma^2)$$

($\gamma = \text{const} \geq 0$). These profiles correspond to a linear velocity shear and the density distribution corresponding to it in the sense of condition (4.1).

Linear velocity shear is a simple example of shear flow and is therefore of special interest. Another example of main flow with a linear shear is also considered for a nearly linear density profile, i.e., for $\rho_1(y) = -y$. In this case, the eigenvalues and eigenfunctions of problem (4.2) [or (2.1)] can be found in implicit form [10]:

$$\lambda_0 = \Big(\frac{\gamma\pi}{\ln(1+\gamma)}\Big)^2 + \frac{\gamma^2}{4}, \qquad \xi_0 = \sqrt{1+\gamma y}\,\sin\Big(\frac{\pi\ln(1+\gamma y)}{\ln(1+\gamma)}\Big).$$

Then, condition (4.1) becomes

$$\frac{\alpha^3(1 + \mathrm{e}^{-\pi/\alpha})}{(1+9\alpha^2)(1+\alpha^2)} = 0, \qquad \alpha = \frac{2\pi}{3\ln(1+\gamma)}.$$

It is easy to show that this condition is satisfied only in the limit as $\gamma \to 0$, which corresponds to the absence of a velocity shear. The case of uniform main flow is analyzed in [3]. In particular, the existence of a small nontrivial branch of conjugate flows is found for the case of linear stratification, which is in satisfactory agreement with the results of the present study.

Conclusions. In the present work, two equivalent forms are obtained for the condition which is necessary for the existence of small smooth branches of flows conjugate to a given shear flow. These conditions *a priori* distinguish classes of main-flow density and velocity profiles from which wave configurations in the form of plateau or smooth-bore type solitary wave can branch off.

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